

# COHOMOLOGY OF COMPLEMENTS OF TORIC ARRANGEMENTS ASSOCIATED TO ROOT SYSTEMS

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**ABSTRACT.** We compute the cohomology of the complement of toric arrangements associated to root systems as representations of the corresponding Weyl groups. Specifically, we develop an algorithm for computing the cohomology of the complement of toric arrangements associated to general root systems and we carry out this computation for the exceptional root systems  $G_2$ ,  $F_4$ ,  $E_6$  and  $E_7$ . We also compute the total cohomology of the complement of the toric arrangement associated to  $A_n$  as a representation of the Weyl group and give a formula for its Poincaré polynomial.

## 1. INTRODUCTION

An *arrangement* is a finite set of closed subvarieties of a variety. Despite their simple definition, arrangements are of interest to a wide range of areas of mathematics such as algebraic geometry, topology, combinatorics, Lie theory and singularity theory.

A *toric arrangement* is an arrangement of codimension one subtori inside a torus. Given a root system  $\Phi$  one can construct an associated toric arrangement  $\mathcal{T}_\Phi$  and the Weyl group  $W$  of  $\Phi$  acts on both  $\mathcal{T}_\Phi$  and its complement  $T_\Phi$ . The goal of this work is to compute the cohomology of  $T_\Phi$  as a representation of  $W$ .

Classically, most attention has been given to arrangements of hyperplanes in an affine space and this work does indeed take its inspiration from the world of hyperplane arrangements. In particular, much of Section 3 consists of toric analogues of results in [10] by Fleischmann and Janiszczak and Section 4 takes its inspiration from the methods used in [9] by Felder and Veselov. The starting point was however in algebraic geometry, more specifically in moduli of curves.

In [13], Looijenga shows that the moduli space  $\mathcal{M}_{3,1}[2]$  of genus three curves with symplectic level 2 structure and one marked point has two natural substrata which are closely related to arrangements of hypertori associated to root systems of type  $E_6$  and  $E_7$ . Thus, it was the pursuit in [2] of computing cohomology of  $\mathcal{M}_{3,1}[2]$  that led to the impending computations.

The results come in two flavors. Firstly, in Section 3 we construct an algorithm for computing the cohomology of the complement of the toric arrangement associated to a general root system  $\Phi$ . This algorithm has been implemented in a **Sage** program, which is available from the author upon request. In Section 5 we give the results of this program for the exceptional root systems  $G_2$ ,  $F_4$ ,  $E_6$  and  $E_7$ . For reasons of computational complexity we were unable to complete the list of exceptional root systems with the final root system  $E_8$  with the computers at hand, but we will do so in future work using high performance computing. Secondly, in Section 4 we compute the total cohomology of the complement of the toric arrangement associated to the root system  $A_n$ . The result is given in Theorem 4.8. We

also prove a toric analogue of Arnold's formula [1] for the Poincaré polynomial of the complement of the arrangement of hyperplanes associated to  $A_n$ . The result is presented in Theorem 4.10.

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## 2. GENERAL ARRANGEMENTS

Unless otherwise specified, we shall always work over the complex numbers.

**Definition 2.1.** Let  $X$  be a variety. An *arrangement*  $\mathcal{A}$  in  $X$  is a finite set  $\{A_i\}_{i \in I}$  of closed subvarieties of  $X$ .

Given an arrangement  $\mathcal{A}$  in a variety  $X$  one may define its *cycle*

$$D_{\mathcal{A}} = \bigcup_{i \in I} A_i \subset X,$$

and its *open complement*

$$X_{\mathcal{A}} = X \setminus D_{\mathcal{A}}.$$

The variety  $X_{\mathcal{A}}$  will be our main object of study.

Many interesting properties of the variety  $X_{\mathcal{A}}$  can be deduced from properties of  $D_{\mathcal{A}}$  via inclusion-exclusion arguments. The object that governs the principle of inclusion and exclusion in this setting is the *intersection poset* of  $\mathcal{A}$ .

**Definition 2.2.** Let  $\mathcal{A}$  be an arrangement in a variety  $X$ . The intersection poset of  $\mathcal{A}$  is the set

$$\mathcal{L}(\mathcal{A}) = \{ \cap_{j \in J} A_j \mid J \subseteq I \}.$$

of intersections of elements of  $\mathcal{A}$ , ordered by inclusion. We include  $X$  as an element of  $\mathcal{L}(\mathcal{A})$  corresponding to the empty intersection.

*Remark 2.3.* The definition of the poset  $\mathcal{L}(\mathcal{A})$  is deceptively similar to a poset used in many combinatorial texts. We therefore point out two key differences. Firstly, the elements of  $\mathcal{L}(\mathcal{A})$  are not necessarily irreducible or even connected. Secondly, combinatorialists usually order their poset by *reverse* inclusion, and this for very good reasons. However, from a geometric viewpoint it is more natural to order  $\mathcal{L}(\mathcal{A})$  by inclusion. Also from a combinatorial perspective there is less reason to order  $\mathcal{L}(\mathcal{A})$  by reverse inclusion. In particular, an interval in  $\mathcal{L}(\mathcal{A})$  is not a geometric lattice regardless of the choice of order.

Since  $\mathcal{L}(\mathcal{A})$  is a poset, it has a Möbius function  $\mu : \mathcal{L}(\mathcal{A}) \times \mathcal{L}(\mathcal{A}) \rightarrow \mathbb{Z}$  defined inductively by setting  $\mu(Z, Z) = 1$  and

$$\sum_{Y \leq Z' \leq Z} \mu(Z', Z) = 0, \quad \text{if } Y \neq Z,$$

where the sum is over all  $Z' \in \mathcal{L}(\mathcal{A})$  between  $Y$  and  $Z$ . Since we shall exclusively be interested in the values of the Möbius function at the maximal element  $X$ , we shall use the simplified notation  $\mu(Z) := \mu(Z, X)$ .

**2.1. Equivariant cohomology.** Let  $\Gamma$  be a finite group of automorphisms of  $X$  that stabilizes  $\mathcal{A}$  as a set. The action of  $\Gamma$  induces actions on  $X_{\mathcal{A}}$  and  $\mathcal{L}(\mathcal{A})$ . For an element  $g \in \Gamma$  we write  $\mathcal{L}^g(\mathcal{A})$  to denote the subposet of  $\mathcal{L}(\mathcal{A})$  consisting of elements which are fixed by  $g$ , and we write  $\mu_g$  to denote the Möbius function of  $\mathcal{L}^g(\mathcal{A})$ .

Given such a group  $\Gamma$ , many questions attains  $\Gamma$ -equivariant counterparts. For instance, consider the following situation. Let  $X$  be a smooth variety over  $\mathbb{C}$ . Such a variety has de Rham cohomology groups  $H^i(X_{\mathcal{A}})$  and compactly supported de Rham cohomology groups  $H_c^i(X_{\mathcal{A}})$  with coefficients in  $\mathbb{Q}$ . The action of  $\Gamma$  on  $X_{\mathcal{A}}$  induces a linear action on both  $H^i(X_{\mathcal{A}})$  and  $H_c^i(X_{\mathcal{A}})$ . In other words, each cohomology group becomes a  $\Gamma$ -representation.

One way to encode this information is via *equivariant Poincaré polynomials*. If  $G$  is a finite group acting on a smooth variety  $Y$  we define the equivariant Poincaré polynomial of  $Y$  at  $g \in G$  as

$$P(Y, t)(g) := \sum_{i \geq 0} \text{Tr}(g, H^i(Y)) \cdot t^i,$$

where  $\text{Tr}(g, H^i(Y))$  denotes the trace of  $g$  on  $H^i(Y)$ . We define the *compactly supported equivariant Poincaré polynomial*  $P_c(Y, t)(g)$  in a completely analogous way.

Poincaré polynomials are not additive so one should not expect inclusion-exclusion arguments to yield formulas for Poincaré polynomials. However, if  $X$  and  $\mathcal{A}$  are nice enough this actually turns out to be the case, e.g. when both  $X$  and  $\mathcal{A}$  are minimally pure. We refer to [7] or [14] for the definition of minimal purity and only note that it is a technical condition concerning the mixed Hodge structure of a variety. This condition is satisfied when  $X$  is an affine or projective space and each element of  $\mathcal{A}$  is a hyperplane (by results of Brieskorn [3]) or when  $X$  is a torus and each element of  $\mathcal{A}$  is a subtorus of codimension one (by results of Looijenga [13]).

**Theorem 2.4** (MacMeican [14]). *Let  $\mathcal{A} = \{A_i\}_{i \in I}$  be a minimally pure arrangement in a minimally pure variety  $X$  and let  $\Gamma$  be a finite group of automorphisms of  $X$  that stabilizes  $\mathcal{A}$  as a set. Then, for each  $g \in \Gamma$*

$$P_c(X_{\mathcal{A}}, t)(g) = \sum_{Z \in \mathcal{L}^g(\mathcal{A})} \mu_g(Z)(-t)^{\text{cd}(Z)} P_c(Z, t)(g),$$

where  $\text{cd}(Z)$  denotes the codimension of  $Z$  in  $X$ .

The proof of Theorem 2.4 is by constructing an Euler characteristic that remembers the Hodge weights and that is additive and therefore allows itself to be computed via an inclusion-exclusion argument. The minimal purity condition then ensures that the Hodge weights are enough to identify the cohomology groups.

**Corollary 2.5.** *Let  $\mathcal{A} = \{A_i\}_{i \in I}$  be a minimally pure arrangement in a minimally pure variety  $X$  and let  $\Gamma$  be a finite group of automorphisms of  $X$  that stabilizes  $\mathcal{A}$  as a set. Suppose also that both  $X_{\mathcal{A}}$  and each element of  $\mathcal{L}(\mathcal{A})$  satisfy Poincaré duality. Then, for each  $g \in \Gamma$*

$$P(X_{\mathcal{A}}, t)(g) = \sum_{Z \in \mathcal{L}^g(\mathcal{A})} \mu_g(Z)(-t)^{\text{cd}(Z)} P(Z, t)(g),$$

where  $\text{cd}(Z)$  denotes the codimension of  $Z$  in  $X$ .

*Proof.* Poincaré duality tells us that if  $M$  is a smooth manifold of complex dimension  $n$ , then (see [15])

$$P_c(M, t) = t^{2n} \cdot P(M, t^{-1}).$$

We apply Poincaré duality to Theorem 2.4 and get

$$t^{2n} P(X_{\mathcal{A}}, t^{-1})(g) = \sum_{Z \in \mathcal{L}^g(\mathcal{A})} \mu_g(Z)(-t)^{\text{cd}(Z)} \cdot t^{2\dim(Z)} \cdot P(Z, t^{-1})(g).$$

We thus have that

$$\begin{aligned} P(X_{\mathcal{A}}, t^{-1})(g) &= \sum_{Z \in \mathcal{L}^g(\mathcal{A})} \mu_g(Z)(-t)^{\text{cd}(Z)} \cdot t^{2\dim(Z)-2n} \cdot P(Z, t^{-1})(g) = \\ &= \sum_{Z \in \mathcal{L}^g(\mathcal{A})} \mu_g(Z)(-t)^{\text{cd}(Z)} \cdot t^{-2\text{cd}(Z)} \cdot P(Z, t^{-1})(g) = \\ &= \sum_{Z \in \mathcal{L}^g(\mathcal{A})} \mu_g(Z)(-t^{-1})^{\text{cd}(Z)} \cdot P(Z, t^{-1})(g). \end{aligned}$$

We now arrive at the desired formula by substituting  $t^{-1}$  for  $t$ .  $\square$

We remark that for arrangements of hyperplanes in an affine space, Corollary 2.5 was first proven by Orlik and Solomon, [17].

**Theorem 2.6** (Looijenga [13]). *Let  $\mathcal{A} = \{A_i\}_{i \in I}$  be an arrangement in a connected variety  $X$  of pure dimension such that  $D_{\mathcal{A}}$  is a divisor which locally can be given as a product of linear functions and such that each element  $Z \in \mathcal{L}(\mathcal{A})$  has pure dimension. Suppose also that both  $X_{\mathcal{A}}$  and each element of  $\mathcal{L}(\mathcal{A})$  satisfy Poincaré duality. Then*

$$E_1^{-p,q} := \bigoplus_{\substack{Z \in \mathcal{L}(\mathcal{A}) \\ \text{cd}(Z)=p}} H^{q-2p}(Z) \otimes_{\mathbb{Z}} \mathbb{Z}^{|\mu(Z)|}(-p),$$

*is a spectral sequence of mixed Hodge structures converging to  $H^{q-p}(X_{\mathcal{A}})$ .*

We refer to [13] for the precise definition of the differentials and only remark that in the cases of interest to us (i.e. hyperplane arrangements and toric arrangements) the spectral sequence degenerates at the  $E_1$ -term.

**2.2. The total cohomology.** Even though Theorem 2.4 is a useful tool, it is often hard to apply if the poset  $\mathcal{L}(\mathcal{A})$  is too complicated. Sometimes it is easier to say something about the action of  $\Gamma$  on the cohomology as a whole (of course, at the expense of getting weaker results). This is the point of view in the following discussion, which is a direct generalization of that of Felder and Veselov in [9].

Let  $\mathcal{A}$  be an arrangement in a variety  $X$  and let  $\Gamma$  be a finite group of automorphisms of  $X$  that fixes  $\mathcal{A}$  as a set. As before,  $\Gamma$  will then act on the individual cohomology groups of  $X_{\mathcal{A}}$  and thus on the total cohomology

$$H^*(X_{\mathcal{A}}) := \bigoplus_{i \geq 0} H^i(X_{\mathcal{A}}).$$

The value of the *total character* at  $g \in \Gamma$  is defined as

$$P(X_{\mathcal{A}})(g) := P(X_{\mathcal{A}}, 1)(g) = \sum_{i \geq 0} \text{Tr}(g, H^i(X_{\mathcal{A}})),$$

and the *Lefschetz number* of  $g \in \Gamma$  is defined as

$$L(X_{\mathcal{A}})(g) := P(X_{\mathcal{A}}, -1)(g) = \sum_{i \geq 0} (-1)^i \cdot \text{Tr}(g, H^i(X_{\mathcal{A}})).$$

Let  $X_{\mathcal{A}}^g$  denote the fixed point locus of  $g \in \Gamma$ . Lefschetz fixed point theorem, see [4], then states that the Euler characteristic  $E(X_{\mathcal{A}}^g)$  of  $X_{\mathcal{A}}^g$  equals the Lefschetz number of  $g$ :

$$E(X_{\mathcal{A}}^g) = L(X_{\mathcal{A}})(g).$$

We now specialize to the case when each cohomology group  $H^i(X_{\mathcal{A}})$  is pure of Tate type  $(i, i)$  and  $\mathcal{A}$  is fixed under complex conjugation. We define an action of  $\Gamma \times \mathbb{Z}_2$  on  $X$  by letting  $(g, 0) \in \Gamma \times \mathbb{Z}_2$  act as  $g \in \Gamma$  and  $(0, 1) \in \Gamma \times \mathbb{Z}_2$  act by complex conjugation. Since  $\mathcal{A}$  is fixed under conjugation, this gives an action on  $X_{\mathcal{A}}$ . We write  $\bar{g}$  to denote the element  $(g, 1) \in \Gamma \times \mathbb{Z}_2$ .

*Remark 2.7.* This action is somewhat different from the action described in [9]. However, it seems that this is the action actually used. The difference is rather small and only affects some minor results.

Since  $H^i(X_{\mathcal{A}})$  has Tate type  $(i, i)$ , complex conjugation acts as  $(-1)^i$  on  $H^i(X_{\mathcal{A}})$ . We thus have

$$\begin{aligned} L(X_{\mathcal{A}})(\bar{g}) &= \sum_{i \geq 0} (-1)^i \cdot \text{Tr}(\bar{g}, H^i(X_{\mathcal{A}})) = \\ &= \sum_{i \geq 0} (-1)^i \cdot (-1)^i \cdot \text{Tr}(g, H^i(X_{\mathcal{A}})) = \\ &= P(X_{\mathcal{A}})(g). \end{aligned}$$

Since  $L(X_{\mathcal{A}})(\bar{g}) = E(X_{\mathcal{A}}^{\bar{g}})$  we have proved the following lemma.

**Lemma 2.8.** *Let  $X$  be a smooth variety and let  $\mathcal{A}$  be an arrangement in  $X$  which is fixed by complex conjugation and such that  $H^i(X_{\mathcal{A}})$  is of pure Tate type  $(i, i)$ . Let  $\Gamma$  be a finite group which acts on  $X$  as automorphisms and which fixes  $\mathcal{A}$  as a set. Then*

$$P(X_{\mathcal{A}})(g) = E(X_{\mathcal{A}}^{\bar{g}}).$$

### 3. TORIC ARRANGEMENTS

By far, the most studied arrangements are arrangements of hyperplanes in affine or projective space. Much of the success of this subject stems from the fact that many computations regarding arrangements of hyperplanes can be carried out solely in terms of the combinatorics of the poset  $\mathcal{L}(\mathcal{A})$ . The last two decades, however, attention has been turned towards the toric analogues. Although toric arrangements share some properties with their hyperplane cousins, the analysis of toric arrangements often require taking also geometrical, topological and arithmetical information into account.

**Definition 3.1.** Let  $X$  be an  $n$ -torus. An arrangement  $\mathcal{A}$  in  $X$  is called a *toric arrangement* if each element of  $\mathcal{A}$  is a hypertorus, i.e. a subtorus of codimension one.

Toric arrangements are also called toral arrangements and arrangements of hypertori.

**Example 3.1.** Let  $X = (\mathbb{C}^*)^2$  and let the arrangement  $\mathcal{A}$  in  $X$  consist of the four subtori given by the equations

$$A_1 : z_1 = 1, \quad A_2 : z_1^2 z_2 = 1, \quad A_3 : z_1 z_2^2 = 1, \quad A_4 : z_2 = 1.$$

Let  $\xi$  be a primitive third root of unity. We then have

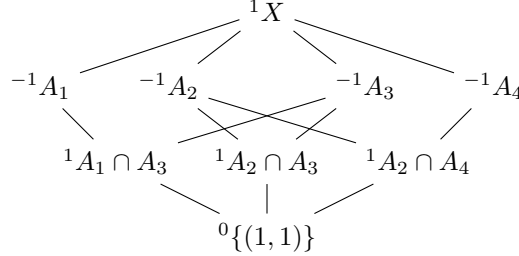
$$A_1 \cap A_2 = A_1 \cap A_4 = A_3 \cap A_4 = \{(1, 1)\},$$

$$A_1 \cap A_3 = \{(1, 1), (1, -1)\},$$

$$A_2 \cap A_3 = \{(1, 1), (\xi, \xi), (\xi^2, \xi^2)\},$$

$$A_2 \cap A_4 = \{(1, 1), (-1, 1)\},$$

and all further intersections are equal to  $\{(1, 1)\}$ . We thus have the poset  $\mathcal{L}(\mathcal{A})$  (the numbers in the upper left corners are the values of the Möbius function):



The Poincaré polynomial of  $X$  is  $(1+t)^2$  and the Poincaré polynomial of  $A_i$  is  $1+t$ . By Corollary 2.5 we now get that the Poincaré polynomial of  $X_{\mathcal{A}}$  is

$$\begin{aligned} P(X_{\mathcal{A}}, t) &= (1+t)^2 + 4 \cdot (-1) \cdot (-t)^1 \cdot (1+t) + (-t)^2 \cdot (2+3+2) = \\ &= 1 + 6t + 12t^2. \end{aligned}$$

**3.1. Toric arrangements associated to root systems.** Let  $\Phi$  be a root system, let  $\Delta = \{\beta_1, \dots, \beta_n\}$  be a set of simple roots and let  $\Phi^+$  be the set of positive roots of  $\Phi$  with respect to  $\Delta$ . We think of  $\Phi$  as a set of vectors in some real Euclidean vector space  $V$  and we let  $M$  be the  $\mathbb{Z}$ -linear span of  $\Phi$ . Thus,  $M$  is a free  $\mathbb{Z}$ -module of finite rank  $n$ .

Each root  $\alpha \in \Phi$  defines a reflection  $r_\alpha$  through the hyperplane perpendicular to it. Explicitly, we have

$$r_\alpha(v) = v - 2 \cdot \frac{\alpha \cdot v}{\alpha \cdot \alpha} \cdot \alpha.$$

These reflections generate the Weyl group  $W$  associated to  $\Phi$ . We remark that, since the roots  $\alpha$  and  $-\alpha$  define the same reflection hyperplane, we have  $r_\alpha = r_{-\alpha}$ .

Define  $T = \text{Hom}(M, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$ . The Weyl group  $W$  acts on  $T$  from the right by precomposition, i.e.

$$(\chi \cdot g)(v) = \chi(g \cdot v).$$

For each  $g \in W$  we define

$$T^g := \{\chi \in T \mid \chi \cdot g = \chi\},$$

and for each  $\alpha \in \Phi$  we define

$$T_\alpha = \{\chi \in T \mid \chi(\alpha) = 1\}.$$

We thus obtain two arrangements of hypertori in  $T$

$$\mathcal{T}^{r_\Phi} = \{T^{r_\alpha}\}_{\alpha \in \Phi}, \quad \text{and} \quad \mathcal{T}_\Phi = \{T_\alpha\}_{\alpha \in \Phi}.$$

Observe that in the definition of  $\mathcal{T}^{r\Phi}$ , we only use reflections and not general group elements. To avoid cluttered notation we shall write  $T^{r\Phi}$  instead of the more cumbersome  $T_{\mathcal{T}^{r\Phi}}$ . Similarly, we write  $T_\Phi$  to mean  $T_{\mathcal{T}_\Phi}$ .

**Lemma 3.2.** *Let  $\alpha$  be an element of  $\Phi$ . The two subtori  $T^{r\alpha}$  and  $T_\alpha$  of  $T$  coincide if and only if the expression*

$$2 \cdot \frac{\alpha \cdot v}{\alpha \cdot \alpha},$$

*takes the value 1 for some  $v \in M$ .*

*Proof.* By definition we have

$$r_\alpha(v) = v - 2 \cdot \frac{\alpha \cdot v}{\alpha \cdot \alpha} \alpha.$$

Hence

$$\chi(r_\alpha(v)) = \frac{\chi(v)}{\chi(\alpha)^{2 \cdot \frac{\alpha \cdot v}{\alpha \cdot \alpha}}},$$

and we thus see that  $\chi(r_\alpha(v)) = \chi(v)$  for all  $v \in M$  if and only if

$$\chi(\alpha)^{2 \cdot \frac{\alpha \cdot v}{\alpha \cdot \alpha}} = 1,$$

for all  $v \in M$ . Hence,  $T_\alpha \subset T^{r\alpha}$  always holds. Also, if  $v$  is such that  $2 \cdot \frac{\alpha \cdot v}{\alpha \cdot \alpha} = 1$  then  $\chi(\alpha)$  must be 1 and it then follows that  $T^{r\alpha} = T_\alpha$ .

On the other hand, if  $2 \cdot \frac{\alpha \cdot v}{\alpha \cdot \alpha} \neq 1$  for all  $v \in M$ , then  $2 \cdot \frac{\alpha \cdot v}{\alpha \cdot \alpha} \in n\mathbb{Z}$  for some integer  $n > 1$ . To see this, assume the contrary, namely that  $2 \cdot \frac{\alpha \cdot v}{\alpha \cdot \alpha} \neq 1$  for all  $v$  but there is no  $n > 1$  which divides  $2 \cdot \frac{\alpha \cdot v}{\alpha \cdot \alpha}$  for all  $v$ . Then there are elements  $v_1$  and  $v_2$  of  $M$  such that

$$n_1 = 2 \cdot \frac{\alpha \cdot v_1}{\alpha \cdot \alpha}, \quad \text{and} \quad n_2 = 2 \cdot \frac{\alpha \cdot v_2}{\alpha \cdot \alpha},$$

are coprime. Let  $a$  and  $b$  be integers such that  $an_1 + bn_2 = 1$ . Then  $av_1 + bv_2 \in M$  and

$$2 \cdot \frac{\alpha \cdot (av_1 + bv_2)}{\alpha \cdot \alpha} = 1.$$

Hence,  $2 \cdot \frac{\alpha \cdot v}{\alpha \cdot \alpha} \in n\mathbb{Z}$  for some integer  $n > 1$ . Thus, the character  $\chi$  which takes the value  $\zeta$ , a primitive  $n$ 'th root of unity, on  $\alpha$  is an element of  $T_{r_\alpha}$  but clearly not an element of  $T_\alpha$ .  $\square$

**Example 3.2.** *We can realize the roots of  $A_n$  as the vectors in  $\mathbb{R}^{n+1}$  of the form  $e_i - e_j$ ,  $i \neq j$ , where  $e_i$  is the  $i$ th coordinate vector. Since  $(e_i - e_j) \cdot (e_i - e_j) = 2$  and  $(e_i - e_j) \cdot (e_i - e_k) = 1$  if  $j \neq k$ , we see that*

$$2 \cdot \frac{(e_i - e_j) \cdot (e_i - e_k)}{(e_i - e_j) \cdot (e_i - e_j)} = 1.$$

*Thus, if  $n > 1$  then every root in  $A_n$  fulfills Lemma 3.2.*

**Example 3.3.** *We can realize the roots of  $B_n$  as the vectors in  $\mathbb{R}^n$  of the form*

$$\begin{aligned} &\pm e_i, \quad i = 1, \dots, n, \\ &e_i - e_j, \quad i \neq j, \\ &\pm(e_i + e_j), \quad i \neq j, \end{aligned}$$

*where  $e_i$  is the  $i$ th coordinate vector. Since  $e_i \cdot e_i = 1$  we have*

$$2 \frac{e_i \cdot v}{e_i \cdot e_i} = 2(e_i \cdot v) \in 2\mathbb{Z},$$

*for all  $v \in M$ . Thus, by Lemma 3.2 we have that  $T^{r_{e_i}} \neq T_{e_i}$ .*

Let  $\chi \in T$ . We introduce the notation  $\chi(\beta_i) = z_i$  for the simple roots  $\beta_i$ ,  $i = 1, \dots, n$ . The coordinate ring of  $T$  is then

$$\mathbb{C}[T] = \mathbb{C}[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}].$$

If  $\alpha$  is a root, there are integers  $m_1, \dots, m_n$  such that

$$\alpha = m_1 \cdot \beta_1 + \dots + m_n \cdot \beta_n.$$

With this notation we have that  $\chi(\alpha) = 1$  if and only if

$$z_1^{m_1} z_2^{m_2} \dots z_n^{m_n} = 1.$$

We denote the Laurent polynomial  $z_1^{m_1} z_2^{m_2} \dots z_n^{m_n} - 1$  by  $f_\alpha$ . Thus,  $\chi$  is an element of  $T_\alpha$  if and only if  $f_\alpha(\chi) = 0$ . If we differentiate  $f_\alpha$  with respect to  $z_i$  we get

$$\frac{\partial f_\alpha}{\partial z_i} = m_i \cdot z_1^{m_1} \dots z_i^{m_i-1} \dots z_n^{m_n},$$

which clearly is nonzero everywhere. Thus, each  $T_\alpha$  is smooth.

Describing the cohomology of  $T_\Phi$  as a  $W$ -representation is a nontrivial task. However, in low cohomological degrees we can say something in general. To begin,  $H^0(T_\Phi)$  is of course always the trivial representation. We can also describe  $H^1(T_\Phi)$  but to do so we need some notation.

**Definition 3.3.** Let  $\Phi$  be a root system of rank  $n$ , realized in a vector space  $V$  of dimension  $n$ .

- (i) The representation given by the action of  $W$  on  $V = M \otimes_{\mathbb{Z}} \mathbb{C}$  is called the *standard representation* and is denoted  $\chi_{\text{std}}$ .
- (ii) The group  $W$  permutes the lines generated by elements of  $\Phi$ . These lines are in bijective correspondence with the positive roots  $\Phi^+$ . We call the resulting permutation representation the *positive representation* and denote it by  $\chi_{\text{pos}}$ .

*Remark 3.4.* The positive representation is positive in two senses. Firstly, it is a permutation representation so it only takes non-negative values. Secondly, it is defined in terms of positive roots.

**Lemma 3.5.** *Let  $\Phi$  be a root system. Then*

$$H^1(T_\Phi) = \chi_{\text{std}} + \chi_{\text{pos}}.$$

*Proof.* Define

$$P = \bigoplus_{\alpha \in \Phi} H^0(T_\alpha)$$

By Theorem 2.6 we have

$$H^1(T_\Phi) = H^1(T) \oplus P.$$

We clearly have  $P = \chi_{\text{pos}}$  and since  $T = \text{Hom}(M, \mathbb{C}^*)$  it follows that  $H^1(T, \mathbb{Z}) = M$ .  $\square$



**3.2. Equivariant cohomology of intersections of hypertori.** Let  $V(f)$  denote the variety defined by  $f$ . A variety  $Z \in \mathcal{L}(\mathcal{T}_\Phi)$  is an intersection

$$Z = \bigcap_{\alpha \in S} T_\alpha = \bigcap_{\alpha \in S} V(f_\alpha),$$

where  $S$  is a subset of  $\Phi$ . We define the ideal

$$I_S = (f_\alpha)_{\alpha \in S} \subseteq \mathbb{C}[T].$$

Then  $Z = V(I_S)$ . The ideal  $I_S$  is entirely determined by the exponents occurring in the various Laurent polynomials  $f_\alpha$  generating it or, in other words, the coefficients occurring in the elements of  $S$  when expressed in terms of the simple roots  $\Delta$ . Thus, if we define the *module of exponents*

$$N_S := \mathbb{Z}\langle S \rangle \subseteq M,$$

then the module  $N_S$  determines  $I_S$  and

$$V(I_S) = \text{Hom}(M/N_S, \mathbb{C}^*) \subseteq \text{Hom}(M, \mathbb{C}^*) = T.$$

For more details, see [8].

Let  $L$  be a free  $\mathbb{Z}$ -module. Recall that the torus  $T_L = \text{Hom}(L, \mathbb{C}^*)$  has cohomology given by

$$H^i(T_L) = \bigwedge^i H^1(T_L) = \bigwedge^i L.$$

Suppose  $L'$  is another free  $\mathbb{Z}$ -module and let  $T_{L'} = \text{Hom}(L', \mathbb{C}^*)$ . A morphism  $L \rightarrow L'$  of free  $\mathbb{Z}$ -modules gives rise to a morphism  $T_{L'} \rightarrow T_L$  and the induced map  $H^i(T_L) \rightarrow H^i(T_{L'})$  is the map

$$\bigwedge^i L \rightarrow \bigwedge^i L'.$$

For these statements, see Chapter 9 of [6].

The modules  $M/N_S$  will not always be free but do still determine the cohomology of  $V(I_S)$  in a sense very similar to the above. Let  $L$  be a free  $\mathbb{Z}$ -module,  $N \subset L$  a submodule and let  $Q = M/L$ . The module  $Q$  will split as a direct sum  $Q = Q^T \oplus Q^F$ , where  $Q^T$  is the torsion part and  $Q^F$  is the free part of  $Q$ . The variety  $T_Q = \text{Hom}(Q, \mathbb{C}^*)$  consists of  $|Q^T|$  connected components, each isomorphic to  $\text{Hom}(Q^F, \mathbb{C}^*)$ . The  $i$ th cohomology group of  $T_Q$  is given by

$$H^i(T_Q) = \bigoplus_{v \in Q^T} \bigwedge^i Q^F.$$

Let  $\varphi : L \rightarrow L'$  be a homomorphism and define  $Q' = L'/\varphi(N)$ . The morphism  $\varphi$  induces a morphism  $Q \rightarrow Q'$  which in turn gives rise to a morphism  $T_{Q'} \rightarrow T_Q$  and the induced map  $H^i(T_Q) \rightarrow H^i(T_{Q'})$  is the map

$$\bigoplus_{v \in Q^T} \bigwedge^i Q^F \rightarrow \bigoplus_{v' \in Q'^T} \bigwedge^i Q'^F.$$

The situation is perhaps clarified by the following. Consider the module  $Q_S = M/N_S$ . The module  $Q_S$  is determined by the echelon basis matrix of  $N_S$ . Torsion elements of  $Q_S$  stems from rows in the echelon basis matrix whose entries has a greatest common divisor greater than 1. The module  $Q_S^F$  is the module  $M/\text{Sat}(N_S)$ , where  $\text{Sat}(N_S) = N_S \otimes_{\mathbb{Z}} \mathbb{Q} \cap M$  is the saturation of  $N_S$ , i.e. the module obtained

from  $N_S$  by dividing each row in the echelon basis matrix of  $N_S$  by the greatest common divisor of its entries.

A row  $(m_1, \dots, m_n)$  in the echelon basis matrix of  $N_S$  corresponds to the equation

$$z_1^{m_1} \dots z_n^{m_n} = 1.$$

If  $\gcd(m_1, \dots, m_n) = d$  we may write  $m_i = d \cdot m'_i$  and

$$(z_1^{m'_1} \dots z_n^{m'_n})^d = 1,$$

i.e. an equation for  $d$  non-intersecting hypertori, namely the hypertorus given by

$$z_1^{m'_1} \dots z_n^{m'_n} = 1,$$

translated by multiplication by powers of a primitive  $d$ th root of unity.

A linear map  $g : M \rightarrow M$  which fixes  $N_S$  can be analyzed in two steps. Firstly, we can investigate how it “permutes different roots of unity”, more precisely, how it acts on  $Q_S^T$ . The elements of  $Q_S^T$  correspond to connected components of  $V(I_S)$  and a component is fixed by  $g$  if and only if the corresponding element of  $Q_S^T$  is fixed. Of course, only fixed components can contribute to the trace of  $g$  on  $H^i(V(I_S))$ . Once we have determined which of the components that are fixed it suffices to compute the trace of  $g$  on the cohomology on one of those components, e.g. the component corresponding to the zero element of  $Q_S^T$ .

We may now write down an algorithm for computing the equivariant Poincaré polynomial of an element  $Z \in \mathcal{L}(\mathcal{T}_\Phi)$ .

**Algorithm 3.6.** *Let  $\Phi$  be a root system of rank  $n$  with Weyl group  $W$  and let  $Z \in \mathcal{L}(\mathcal{T}_\Phi)$  correspond to the subset  $S$  of  $\Phi$ . Let  $g$  be an element of  $W$  stabilizing  $Z$ . Then  $P(Z, t)(g)$  can be computed via the following steps.*

- (1) *Compute the number  $m$  of elements in  $Q_S^T$  which are fixed by  $g$  (for instance by lifting each element of  $Q_S^T$  to  $M$ , acting on the lifted element by  $g$  and pushing the result down to  $Q_S^T$ ).*
- (2) *Compute  $\text{Tr}(g, Q_S^F)$  (for instance as  $\text{Tr}(g, M) - \text{Tr}(g, N_S)$ ).*
- (3) *Using the knowledge of  $\text{Tr}(g, Q_S^F)$ , compute  $\text{Tr}(g, \wedge^i Q_S^F)$  for  $i = 1, \dots, n$  (for instance via the Newton-Girard method).*

The polynomial  $P(Z, t)(g)$  is now given by

$$P(Z, t)(g) = m \cdot \sum_{i=0}^n \text{Tr}(g, \wedge^i Q_S^F) t^i.$$

**3.3. Posets of hypertoric arrangements associated to root systems.** Section 3.2 tells us how to compute  $\text{Tr}(g, H^i(Z))$  for any  $Z \in \mathcal{L}(\mathcal{T}_\Phi)$  and thus, via Poincaré duality, how to compute  $\text{Tr}(g, H_c^i(Z))$ . However, in order to use Corollary 2.5 to compute  $P(\mathcal{T}_\Phi, t)(g)$  we also need to compute the poset  $\mathcal{L}^g(\mathcal{T}_\Phi)$ . The following discussion takes its inspiration from Fleischmann and Janiszczak, [10].

We have seen that each element  $Z = \cap_{\alpha \in S} T_\alpha$  is given by its module of exponents  $N_S = \mathbb{Z}\langle S \rangle \subset M$ . We may therefore equally well investigate the modules  $N_S$ . However, an inclusion  $N_S \subset N_{S'}$  gives a surjection  $M/N_S \twoheadrightarrow M/N_{S'}$  and thus an inclusion  $\text{Hom}(M/N_{S'}, \mathbb{C}^*) \hookrightarrow \text{Hom}(M/N_S, \mathbb{C}^*)$ . Hence, the poset structure should be given by reverse inclusion.

**Definition 3.7.** Let  $\Phi$  be a root system. The poset of modules of exponents is the set

$$\mathcal{P}(\Phi) = \{\mathbb{Z}\langle S \rangle \mid S \subseteq \Phi\},$$

ordered by reverse inclusion. If  $g$  is an element of the Weyl group of  $\Phi$ , we write  $\mathcal{P}^g(\Phi)$  to denote the subposet of  $\mathcal{P}(\Phi)$  of modules fixed by  $g$ .

By construction we have that the poset  $\mathcal{P}^g(\Phi)$  is isomorphic to the poset  $\mathcal{L}^g(\mathcal{T}_\Phi)$ . The benefit of considering the posets  $\mathcal{P}^g(\Phi)$  instead is that they are more easily computed.

Let  $g$  be an element of the Weyl group of  $\Phi$ . If  $N$  is an element of  $\mathcal{P}^g(\Phi)$ , then  $N \cap \Phi$  is a union of  $g$ -orbits of  $\Phi$ . Since  $N = \mathbb{Z}\langle N \cap \Phi \rangle$ , we may compute  $\mathcal{P}^g(\Phi)$  via the following steps.

**Algorithm 3.8.** *Let  $\Phi$  be a root system and let  $W$  be its Weyl group. Let  $g$  be an element of  $g$ . Then the following algorithm computes the poset  $\mathcal{P}^g(\Phi)$ :*

- (1) *Compute the  $g$ -orbits of  $\Phi$ .*
- (2) *Compute the set  $\mathcal{P}^g(\Phi)_{\text{set}}$  of all (distinct)  $\mathbb{Z}$ -spans of unions of  $g$ -orbits.*
- (3) *Investigate the inclusion relations of the elements of  $\mathcal{P}^g(\Phi)_{\text{set}}$ .*

Using Algorithms 3.6 and 3.8 we have constructed a **Sage** program computing the equivariant Poincaré polynomials of the complement of a toric arrangement associated to a root system  $\Phi$ . This program has been used to compute these polynomials for many root systems of small rank. In Section 5 we give the results for all exceptional root system except for  $E_8$ . At the moment the  $E_8$ -case seems beyond reach for an ordinary computer but we will complete the computation also for the  $E_8$  case, using high performance computing.

In practice, we represent elements of  $\mathcal{P}^g(\Phi)_{\text{set}}$  by their echelon basis matrices (which need to be computed with some care since we are working over the integers), so step (2) consists of performing Gaussian elimination on matrices and making sure we only save each matrix once. Step (3) is by far the most computationally demanding since, in principle, one needs to make  $|\mathcal{P}^g(\Phi)|^2$  comparisons. Fortunately, it is very parallelizable.

**3.4. Posets of hyperplane arrangements associated to root systems.** Given a root system  $\Phi$  we can also define a corresponding hyperplane arrangement. Let  $V = \text{Hom}(M, \mathbb{C}) \cong \mathbb{C}^n$  and for each  $\alpha \in \Phi$  define

$$V_\alpha = \{\phi \in V \mid \phi(\alpha) = 0\}.$$

We thus obtain a hyperplane arrangement in  $V$

$$\mathcal{V}_\Phi = \{V_\alpha\}_{\alpha \in \Phi}.$$

As usual, we denote the corresponding poset by  $\mathcal{L}(\mathcal{V}_\Phi)$ .

We can analyze  $\mathcal{L}(\mathcal{V}_\Phi)$  in a way analogous to how we analyzed  $\mathcal{L}(\mathcal{T}_\Phi)$ . For instance, if we take  $\mathbb{C}$ -spans instead of taking  $\mathbb{Z}$ -spans in step (2) of Algorithm 3.8 we obtain a smaller poset  $\mathcal{A}^g(\Phi)$  which is isomorphic to  $\mathcal{L}(\mathcal{V}_\Phi)$ . We have a order preserving map

$$(3.1) \quad \tau : \mathcal{P}^g(\Phi) \rightarrow \mathcal{A}^g(\Phi),$$

sending a module  $N$  to  $N \otimes_{\mathbb{Z}} \mathbb{C}$ . Note that  $\tau$  sends a module of rank  $r$  to a vector space of dimension  $r$ . In terms of echelon basis matrices,  $\tau$  sends a echelon basis matrix to its saturation. Thus, the map  $\tau$  is an isomorphism of posets if and only if each module of  $\mathcal{P}^g(\Phi)$  is saturated.

4. THE TORIC ARRANGEMENT ASSOCIATED TO  $A_n$ 

The simplest of the irreducible root systems are the root systems of type  $A_n$ . The root system  $A_n$  is most naturally viewed as an  $n$ -dimensional subspace of  $\mathbb{R}^{n+1}$ . Denote the  $i$ th coordinate vector of  $\mathbb{R}^{n+1}$  by  $e_i$ . The roots  $\Phi$  can then be chosen to be

$$\alpha_{i,j} = e_i - e_j, \quad i \neq j.$$

A choice of positive roots is

$$\alpha_{i,j} = e_i - e_j, \quad i < j,$$

and the simple roots with respect to this choice of positive roots are

$$\beta_i = e_i - e_{i+1}, \quad i = 1, \dots, n.$$

The Weyl group of group  $A_n$  is isomorphic to the symmetric group  $S_{n+1}$  and an element of  $S_{n+1}$  acts on an element in  $M = \mathbb{Z}\langle\Phi\rangle$  by permuting the indices of the coordinate vectors in  $\mathbb{R}^{n+1}$ .

**4.1. The total character.** In this section we shall compute the total character of any element  $g \in A_n$ . This will determine the total cohomology  $H^*(T_\Phi)$  as an  $A_n$ -representation. Although we have not pursued this, similar methods should allow the computation of  $H^*(T_\Phi)$  also in the case of root systems of type  $B_n$ ,  $C_n$  and  $D_n$ .

**Lemma 4.1.** *Let  $W$  be the Weyl group of  $A_n$  and suppose that  $g \in W$  has a cycle of length greater than two. Then  $T_\Phi^g$  is empty.*

*Proof.* The statement only depends on the conjugacy class of  $g$  so suppose that  $g$  contains the cycle  $(12\dots s)$ , where  $s \geq 3$ . We then have

$$\begin{aligned} g.\beta_1 &= e_2 - e_3 = \beta_2, \\ g.\beta_2 &= e_3 - e_4 = \beta_3, \\ &\vdots \\ g.\beta_{s-2} &= e_{s-1} - e_s = \beta_{s-1}, \\ g.\beta_{s-1} &= e_s - e_1 = -(\beta_1 + \dots + \beta_{s-1}). \end{aligned}$$

If  $\bar{g}.\chi = \chi$  we must have

$$\begin{aligned} (1) \quad z_i &= \bar{z}_{i+1} \quad \text{for } i = 1, \dots, s-2, \\ (2) \quad z_{s-1} &= \bar{z}_1^{-1} \bar{z}_2^{-1} \dots \bar{z}_{s-1}^{-1}. \end{aligned}$$

We insert (1) into (2) and take absolute values to obtain  $|z_1|^s = 1$ . We thus see that  $|z_1| = 1$ . Since we have  $z_2 = \bar{z}_1$  it follows that

$$\chi(\alpha_{1,3}) = \chi(\beta_1 + \beta_2) = z_1 \cdot z_2 = z_1 \cdot \bar{z}_1 = |z_1|^2 = 1.$$

Thus,  $\chi$  lies in  $T_{\alpha_{13}}$  so  $T_\Phi^g$  is empty.  $\square$

**Corollary 4.2.** *Let  $\Phi$  be a root system of rank  $n$  containing  $A_n$  as a subroot system so that the Weyl group of  $\Phi$  contains the Weyl group of  $A_n$  as a subgroup. If  $g$  is an element of the Weyl group of  $A_n$  of order greater than 2, then  $T_\Phi^g$  is empty.*

*Proof.* Since  $A_n$  is a subroot system of  $\Phi$  we have  $\mathcal{T}_{A_n} \subseteq \mathcal{T}_\Phi$ . Hence,  $T_\Phi \subseteq T_{A_n}$  so the Corollary follows since  $T_{A_n}^g = \emptyset$ .  $\square$

We point out that Corollary 4.2 together with Lemma 2.8 provides a test for the program from Section 3 for quite a wide range of root systems. In particular, if we apply Lemma 2.8 to Lemma 4.1 we obtain the following corollary.

**Corollary 4.3.** *If  $g$  is an element of the Weyl group of  $A_n$  such that  $g^2 \neq \text{id}$ , then*

$$P(T_\Phi)(g) = 0.$$

We thus know the total character of all elements in the Weyl group of  $A_n$  of order greater than 2. We shall therefore turn our attention to the involutions.

**Lemma 4.4.** *If  $g$  is an element of the Weyl group of  $A_n$  of order 2 which is not a reflection, then*

$$E(T_\Phi^g) = P(T_\Phi)(g) = 0.$$

*Proof.* Let  $k > 1$  and consider the element  $g = (1\ 2)(3\ 4) \cdots (2k-1\ 2k)$ . We define a new basis for  $M$ :

$$\begin{aligned} \gamma_i &= \beta_i + \beta_{i+1}, & i &= 1, \dots, 2k-2, \\ \gamma_j &= \beta_j & j &= 2k-1, \dots, n \end{aligned}$$

Then

$$g \cdot \gamma_{2i-1} = \gamma_{2i}, \quad \text{and} \quad g \cdot \gamma_{2i} = \gamma_{2i-1},$$

for  $i = 1, \dots, k-1$ ,

$$g \cdot \gamma_{2k-1} = -\gamma_{2k-1}, \quad g \cdot \gamma_{2k} = \gamma_{2k-1} + \gamma_{2k},$$

and  $g \cdot \gamma_i = \gamma_i$  for  $i > 2k$ . If we put  $\chi(\gamma_i) = t_i$ , then  $T_\Phi^g \subseteq T_\Phi$  is given by the equations

$$\begin{aligned} t_{2i-1} &= \bar{t}_{2i} & i &= 1, \dots, k-1, \\ t_{2k-1} &= \bar{t}_{2k-1}^{-1}, \\ t_{2k} &= \bar{t}_{2k-1} \cdot \bar{t}_{2k}, \\ t_i &= \bar{t}_i, & i &= 2k+1, \dots, n. \end{aligned}$$

Thus, the points of  $T_\Phi^g$  have the form

$$(t_1, \bar{t}_1, t_2, \bar{t}_2, \dots, t_{k-1}, \bar{t}_{k-1}, s, s^{-1/2} \cdot r, t_{2k}, \dots, t_n),$$

where  $s \in S^1 \setminus \{1\} \subset \mathbb{C}$ ,  $r \in \mathbb{R}^+$  and  $t_i \in \mathbb{R}$  for  $i = 2k, \dots, n$ .

We can now see that each connected component of  $T_\Phi^g$  is homeomorphic to  $\mathcal{C}_{k-1}(\mathbb{C} \setminus \{0, 1\}) \times (0, 1) \times \mathbb{R}^{n-2k+1}$ , where  $\mathcal{C}_{k-1}(\mathbb{C} \setminus \{0, 1\})$  is the configuration space of  $k-1$  points in the twice punctured complex plane. This space is in turn homotopic to the configuration space  $\mathcal{C}_{k+1}(\mathbb{C})$  of  $k+1$  points in the complex plane. The space  $\mathcal{C}_{k+1}(\mathbb{C})$  is known to have Euler characteristic zero for  $k \geq 1$ .  $\square$

*Remark 4.5.* Note that it is essential that we use ordinary cohomology in the above proof, since compactly supported cohomology is not homotopy invariant. However, since  $T_\Phi$  satisfies Poincaré duality, the corresponding result follows also in the compactly supported case.

We now turn to the reflections.

**Lemma 4.6.** *If  $g$  is a reflection in the Weyl group of  $A_n$ , then*

$$E(T_\Phi^g) = P(T_\Phi)(g) = n!.$$

*Proof.* Let  $g = (12)$ . We then have

$$\begin{aligned} g.\beta_1 &= -\beta_1, \\ g.\beta_2 &= \beta_1 + \beta_2, \\ g.\beta_i &= \beta_i, \end{aligned} \quad i > 2.$$

This gives the equations

$$\begin{aligned} z_1 &= \bar{z}_1^{-1}, \\ z_2 &= \bar{z}_1 \cdot \bar{z}_2, \\ z_i &= \bar{z}_i, \end{aligned} \quad i > 2.$$

Thus  $z_1 \in S^1 \setminus \{1\}$ ,  $z_2$  is not real and satisfies  $z_2 = \bar{z}_1 \cdot \bar{z}_2$  so we choose  $z_2$  from a space isomorphic to  $\mathbb{R}^*$ . Hence,  $T_\Phi^g \cong [0, 1] \times \mathbb{R}^* \times M$  where  $M$  is the space where the last  $n - 2$  coordinates  $z_3, \dots, z_n$  takes their values.

These coordinates satisfy  $z_i = \bar{z}_i$ , i.e. they are real. We begin by choosing  $z_3$ . Since  $\chi(e_i - e_j) \neq 0, 1$  we need  $z_3 \neq 0, 1$ . We thus choose  $z_3$  from  $\mathbb{R} \setminus \{0, 1\}$ . We then choose  $z_4$  in  $\mathbb{R} \setminus \{0, 1, \frac{1}{z_3}\}$ ,  $z_5$  in  $\mathbb{R} \setminus \{0, 1, \frac{1}{z_4}, \frac{1}{z_3 \cdot z_4}\}$  and so on. In the  $i$ th step we have  $i$  components to choose from. Thus,  $M$  consists of

$$3 \cdot 4 \cdots n = \frac{n!}{2},$$

components, each isomorphic to  $\mathbb{R}^{n-2}$ . Hence,  $E(M) = \frac{n!}{2}$  and it follows that

$$E(T_\Phi^g) = E([0, 1]) \cdot E(\mathbb{R}^*) \cdot E(M) = n!.$$

□

It remains to compute the value of the total character at the identity element.

**Lemma 4.7.**  $E(T_\Phi^{\text{id}}) = P(T_\Phi)(\text{id}) = \frac{(n+2)!}{2}$

*Proof.* The proof is a calculation similar to that in the proof of Lemma 4.6. We note that the equations for  $T_\Phi^{\text{id}}$  are  $z_i = \bar{z}_i$ ,  $i = 1, \dots, n$  so the computation of  $E(T_\Phi^{\text{id}})$  is essentially the same as that for  $E(M)$  above. The difference is that we have  $n$  steps and in the  $i$ 'th step we have  $i + 1$  choices. This gives the result. □

Let  $W$  denote the Weyl group of  $A_n$ . Lemmas 4.1, 4.4, 4.6 and 4.7 together determine the character of  $W$  on  $H^*(T_\Phi)$ . The corresponding calculation for the affine hyperplane case was first computed by Lehrer in [12] and later by Felder and Veselov in [9]. In the hyperplane case, the total cohomology turned out to be  $2 \text{Ind}_{\langle s \rangle}^W(\text{Triv}_{\langle s \rangle})$ , where  $s$  is a transposition, i.e. the cohomology is twice the representation induced up from the trivial representation of the subgroup generated by a transposition. It turns out that the representation  $\text{Ind}_{\langle s \rangle}^W(\text{Triv}_{\langle s \rangle})$  accounts for most of the cohomology also in the toric case.

The representation  $\text{Ind}_{\langle s \rangle}^W(\text{Triv}_{\langle s \rangle})$  takes value  $(n + 1)!$  on the identity,  $2(n - 1)!$  on transpositions and is zero elsewhere. Since the character of  $H^*(T_\Phi)$  takes the value  $(n + 2)!/2$  on the identity,  $n!$  on transpositions and is zero elsewhere we can now see the following.

**Theorem 4.8.** *Let  $\Phi$  be the root system  $A_n$  and let  $W$  be the Weyl group of  $A_n$ . Then the total cohomology of  $T_\Phi$  is the  $W$ -representation*

$$H^*(T_\Phi) = \text{Reg}_W + n \cdot \text{Ind}_{\langle s \rangle}^W(\text{Triv}_{\langle s \rangle}),$$

where  $\text{Reg}_W$  is the regular representation of  $W$  and  $\text{Ind}_{\langle s \rangle}^W(\text{Triv}_{\langle s \rangle})$  denotes the representation of  $W$  induced up from the trivial representation of the subgroup generated by the simple reflection  $s = (12)$ .

**4.2. The Poincaré polynomial.** In [1] Arnold proved that the Poincaré polynomial of the complement of the affine hyperplane arrangement associated to  $A_n$  is

$$(4.1) \quad \prod_{i=1}^n (1 + i \cdot t).$$

This result was later reproved and generalized by Orlik and Solomon in [17]. In this section we shall see that the Poincaré polynomial of the complement of the toric arrangement associated to  $A_n$  satisfies a similar formula.

Recall the map  $\tau : \mathcal{P}(\Phi) \rightarrow \mathcal{A}(\Phi)$  defined in Equation (3.1). The map  $\tau$  is an isomorphism if and only if each module of  $\mathcal{P}(\Phi)$  is saturated. For  $\Phi = A_n$  this is indeed the case. The pivotal observation for proving this is the following lemma, which can be proven via a simple induction argument on the number of rows using Gaussian elimination.

**Lemma 4.9.** *Let  $A$  be a binary matrix of full rank such that the 1's in each row of  $A$  are consecutive. Then each pivot element in the row reduced echelon matrix (over  $\mathbb{Z}$ ) obtained from  $A$  is 1.*

If one expresses the roots of  $A_n$  in terms of the simple roots  $\beta_i$ , each root is a vector of zeros and ones with all ones consecutive. It thus follows from Lemma 4.9 that the modules in  $\mathcal{P}(\Phi)$  are saturated. Hence, the map  $\tau : \mathcal{P}(\Phi) \rightarrow \mathcal{A}(\Phi)$  is an isomorphism of posets. This is a key ingredient in proving the following formula.

**Theorem 4.10.** *Let  $\Phi$  be the root system  $A_n$ . The Poincaré polynomial of  $T_\Phi$  is given by*

$$P(T_\Phi, t) = \prod_{i=1}^n (1 + (i+1) \cdot t).$$

*Proof.* Let  $\mu_{\mathcal{A}}$  denote the Möbius function of  $\mathcal{A}(\Phi)$ . Theorem 2.4 and Equation (4.1) tell us that

$$\sum_{V \in \mathcal{A}(\Phi)} \mu_{\mathcal{A}}(V) \cdot (-t)^{\dim(V)} = \prod_{i=1}^n (1 + i \cdot t).$$

By equating the coefficients of  $t^r$  we get

$$\begin{aligned} \sum_{\substack{V \in \mathcal{A}(\Phi) \\ \dim(V)=r}} \mu_{\mathcal{A}}(V) (-1)^r &= \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=r}} \prod_{i \in I} i = \\ &= e_r(1, \dots, n), \end{aligned}$$

where  $e_r$  denotes the  $r$ th elementary symmetric polynomial. We saw above that the map  $\tau : \mathcal{P}(\Phi) \rightarrow \mathcal{A}(\Phi)$  is an isomorphism of posets. Hence, if  $\mu_{\mathcal{P}}$  denotes the Möbius function of  $\mathcal{P}(\Phi)$  we have that

$$\mu_{\mathcal{P}}(N) = \mu_{\mathcal{A}}(\tau(N)).$$

It thus follows that

$$(4.2) \quad \sum_{\substack{N \in \mathcal{P}(\Phi) \\ \text{rk}(N)=r}} \mu_{\mathcal{P}}(N)(-1)^r = e_r(1, \dots, n).$$

If we apply Theorem 2.4 to  $T_{\Phi}$ , we obtain

$$\begin{aligned} P(T_{\Phi}, t) &= \sum_{N \in \mathcal{P}(\Phi)} \mu_{\mathcal{P}}(N)(-t)^{\text{rk}(N)} \cdot (1+t)^{n-\text{rk}(N)} \\ &= \sum_{r=0}^n t^r \cdot (1+t)^{n-r} \sum_{\text{rk}(N)=r} \mu_{\mathcal{P}}(N) \cdot (-1)^r. \end{aligned}$$

If we use Equation (4.2) we now see that the coefficient of  $t^k$  in  $P(T_{\Phi}, t)$  is

$$\sum_{j=0}^k \binom{n-j}{k-j} \cdot e_j(1, \dots, n).$$

The coefficient of  $t^k$  in  $\prod_{i=1}^n (1 + (i+1) \cdot t)$  is

$$\begin{aligned} \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=k}} (i_1+1) \cdots (i_k+1) &= \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=k}} \sum_{j=0}^k e_j(i_1, \dots, i_k) = \\ &= \sum_{j=0}^k \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=k}} e_j(i_1, \dots, i_k) = \\ &= \sum_{j=0}^k \binom{n-j}{k-j} \cdot e_j(1, \dots, n). \end{aligned}$$

This proves the claim.  $\square$

Note that setting  $t = 1$  in Theorem 4.10 gives another proof of Lemma 4.7.

## 5. TORIC ARRANGEMENTS ASSOCIATED TO EXCEPTIONAL ROOT SYSTEMS

As mentioned before, we have used Algorithms 3.6 and 3.8 to construct a **Sage** program computing the equivariant cohomology of the complement of a toric arrangement associated to a root system. One application of the results in Section 4 has been to verify the validity of this program. Below, we also compare our results with earlier work. In this section we present the results produced by this program for the exceptional root systems  $G_2$ ,  $F_4$ ,  $E_6$  and  $E_7$ . This only leaves the exceptional root system  $E_8$ . We will complete the list with this case in future work.

The irreducible representations of Weyl groups of exceptional root systems can be described in terms of two integers,  $d$  and  $e$ . The integer  $d$  is the degree of the representation. The integer  $e$  can be defined as follows, see [5], p. 411.

**Definition 5.1.** Let  $\Phi$  be an exceptional root system and let  $W$  be its Weyl group. Then each irreducible representation of  $W$  occurs in some symmetric power  $\text{Sym}^i \chi_{\text{std}}$  of  $\chi_{\text{std}}$ . Given an irreducible representation  $\chi$ , let  $e$  be the integer such that  $\chi$  occurs as a direct summand in  $\text{Sym}^e \chi_{\text{std}}$  but not in  $\text{Sym}^i \chi_{\text{std}}$  for all  $i < e$ .



For root systems of type  $E$ , the integers  $d$  and  $e$  uniquely determine the irreducible representations. However, for the root systems  $F_4$  and  $G_2$  this is not the case. We denote the irreducible character corresponding to  $d$  and  $e$  by  $\phi_d^e$ . If there are two characters corresponding to the same  $d$  and  $e$  we add a second subscript to distinguish between them.

We refer to [5] for character tables of the Weyl groups of type  $G_2$  and  $F_4$ . It should be noted that Carter denotes the characters  $\phi_d^e$ ,  $\phi_{d,1}^e$  and  $\phi_{d,2}^e$  by  $\phi_{d,e}$ ,  $\phi'_{d,e}$  and  $\phi''_{d,e}$ . We have chosen to denote the characters differently because we are in need of notational compactness.

**5.1. The root system  $G_2$ .** Let  $\Phi$  be the root system of type  $G_2$ . Then the Poincaré polynomial of  $T_\Phi$  is

$$P(T_\Phi, t) = 19t^2 + 8t + 1.$$

The cohomology of  $T_\Phi$  as a representation of the Weyl group of  $G_2$  is given in Table 5.1.

	$\phi_1^0$	$\phi_1^6$	$\phi_{1,1}^3$	$\phi_{1,2}^3$	$\phi_2^1$	$\phi_2^2$
$H^0(T_\Phi)$	1	0	0	0	0	0
$H^1(T_\Phi)$	2	0	0	0	1	2
$H^2(T_\Phi)$	2	1	1	1	3	4

TABLE 1. The cohomology groups of the complement of the toric arrangement associated to  $G_2$  as representations of the Weyl group.

**5.2. The root system  $F_4$ .** Let  $\Phi$  be the root system of type  $F_4$ . Then the Poincaré polynomial of  $T_\Phi$  is

$$P(T_\Phi, t) = 2153t^4 + 1260t^3 + 286t^2 + 28t + 1.$$

The cohomology of  $T_\Phi$  as a representation of the Weyl group of  $G_2$  is given in Table 5.2.

*Remark 5.2.* The polynomial  $P(T_\Phi, t)$  was computed by Moci in [16] as the Poincaré polynomial of the complement of the toric arrangement associated to the *coroot* system of  $F_4$ .

**5.3. The root system  $E_6$ .** Let  $\Phi$  be the root system of type  $E_6$ . Then the Poincaré polynomial of  $T_\Phi$  is

$$P(T_\Phi, t) = 58555t^6 + 63378t^5 + 27459t^4 + 6020t^3 + 705t^2 + 42t + 1.$$

The cohomology of  $T_\Phi$  as a representation of the Weyl group of  $E_6$  is given in Table 5.3.

*Remark 5.3.* The column  $\phi_1^0$  gives the cohomology of  $T_\Phi/W$ . It was first computed by Looijenga in [13].

	$\phi_1^0$	$\phi_1^{24}$	$\phi_{1,1}^{12}$	$\phi_{1,2}^{12}$	$\phi_{2,1}^{16}$	$\phi_{2,2}^4$	$\phi_{2,2}^{16}$	$\phi_{2,4}^4$	$\phi_4^1$	$\phi_{4,2}^7$
$H^0(T_\Phi)$	1	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	2	0	0	0	0	1	0	1	1	0
$H^2(T_\Phi)$	2	0	0	0	0	3	0	3	2	0
$H^3(T_\Phi)$	3	0	1	2	1	6	2	5	3	3
$H^4(T_\Phi)$	3	1	2	3	3	6	4	5	6	7
	$\phi_{4,1}^7$	$\phi_4^{13}$	$\phi_4^8$	$\phi_{6,2}^6$	$\phi_{6,1}^6$	$\phi_{8,1}^3$	$\phi_{8,2}^9$	$\phi_{8,2}^3$	$\phi_{8,1}^9$	$\phi_9^{10}$
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^2(T_\Phi)$	0	0	2	1	1	2	0	2	0	0
$H^3(T_\Phi)$	2	1	8	7	9	7	5	8	4	7
$H^4(T_\Phi)$	6	5	10	12	14	13	13	14	12	16
	$\phi_{9,1}^6$	$\phi_{9,2}^6$	$\phi_9^2$	$\phi_{12}^4$	$\phi_{16}^5$					
$H^0(T_\Phi)$	0	0	0	0	0					
$H^1(T_\Phi)$	0	0	2	0	0					
$H^2(T_\Phi)$	3	4	9	3	2					
$H^3(T_\Phi)$	12	15	20	16	12					
$H^4(T_\Phi)$	18	20	22	25	26					

TABLE 2. The cohomology groups of the complement of the toric arrangement associated to  $F_4$  as representations of the Weyl group.

	$\phi_1^0$	$\phi_1^{36}$	$\phi_6^{25}$	$\phi_6^1$	$\phi_{10}^9$	$\phi_{15}^{17}$	$\phi_{15}^{16}$	$\phi_{15}^5$	$\phi_{15}^4$	$\phi_{20}^{20}$
$H^0(T_\Phi)$	1	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	1	0	0	1	0	0	0	0	1	0
$H^2(T_\Phi)$	0	0	0	1	0	0	0	1	2	0
$H^3(T_\Phi)$	0	0	0	1	1	0	0	5	4	0
$H^4(T_\Phi)$	0	0	0	5	7	4	3	12	10	2
$H^5(T_\Phi)$	1	0	4	10	12	15	14	20	19	17
$H^6(T_\Phi)$	2	1	6	8	12	16	18	17	19	21
	$\phi_{20}^2$	$\phi_{20}^{10}$	$\phi_{24}^{12}$	$\phi_{24}^6$	$\phi_{30}^{15}$	$\phi_{30}^3$	$\phi_{60}^{11}$	$\phi_{60}^5$	$\phi_{60}^8$	$\phi_{64}^{13}$
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	1	0	0	0	0	0	0	0	0	0
$H^2(T_\Phi)$	3	0	0	0	0	2	0	2	1	0
$H^3(T_\Phi)$	6	2	1	4	0	9	3	11	6	1
$H^4(T_\Phi)$	15	11	10	16	9	24	23	38	30	21
$H^5(T_\Phi)$	30	24	26	31	30	41	68	80	74	69
$H^6(T_\Phi)$	25	23	27	29	33	36	66	69	69	69
	$\phi_{64}^4$	$\phi_{80}^7$	$\phi_{81}^6$	$\phi_{81}^{10}$	$\phi_{90}^8$					
$H^0(T_\Phi)$	0	0	0	0	0					
$H^1(T_\Phi)$	0	0	0	0	0					
$H^2(T_\Phi)$	3	0	2	0	0					
$H^3(T_\Phi)$	14	9	14	5	10					
$H^4(T_\Phi)$	45	45	50	36	50					
$H^5(T_\Phi)$	88	99	103	94	111					
$H^6(T_\Phi)$	74	91	92	90	101					

TABLE 3. The cohomology groups of the complement of the toric arrangement associated to  $E_6$  as representations of the Weyl group.

5.4. **The root system  $E_7$ .** Let  $\Phi$  be the root system of type  $E_7$ . Then the Poincaré polynomial of  $T_\Phi$  is

$$P(T_\Phi, t) = 3842020t^7 + 3479734t^6 + 1328670t^5 + 268289t^4 + 30800t^3 + 2016t^2 + 70t + 1.$$

The cohomology of  $T_\Phi$  as a representation of the Weyl group of  $E_7$  is given in Table 5.4.

*Remark 5.4.* The column  $\phi_1^0$  gives the cohomology of  $T_\Phi/W$ . It was first computed by Looijenga in [13] and later corrected by Getzler and Looijenga in [11].

	$\phi_1^0$	$\phi_1^{63}$	$\phi_7^{46}$	$\phi_7^1$	$\phi_{15}^7$	$\phi_{15}^{28}$	$\phi_{21}^{36}$	$\phi_{21}^3$	$\phi_{21}^{33}$	$\phi_{21}^6$
$H^0(T_\Phi)$	1	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	1	0	0	1	0	0	0	0	0	0
$H^2(T_\Phi)$	0	0	0	1	0	0	0	1	0	1
$H^3(T_\Phi)$	0	0	0	0	1	0	0	2	0	3
$H^4(T_\Phi)$	0	0	0	1	4	0	0	3	0	7
$H^5(T_\Phi)$	0	0	0	3	7	3	2	11	2	17
$H^6(T_\Phi)$	2	0	4	9	14	18	19	26	16	34
$H^7(T_\Phi)$	2	1	8	10	21	19	25	30	23	34
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	$\phi_{27}^2$	$\phi_{27}^{37}$	$\phi_{35}^{31}$	$\phi_{35}^4$	$\phi_{35}^{22}$	$\phi_{35}^{13}$	$\phi_{56}^{30}$	$\phi_{56}^3$	$\phi_{70}^{18}$	$\phi_{70}^9$
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	1	0	0	1	0	0	0	0	0	0
$H^2(T_\Phi)$	2	0	0	2	0	0	0	1	0	0
$H^3(T_\Phi)$	3	0	0	3	0	1	0	3	0	1
$H^4(T_\Phi)$	8	0	0	9	2	4	0	9	5	6
$H^5(T_\Phi)$	25	1	3	30	16	14	11	30	30	30
$H^6(T_\Phi)$	50	16	23	63	45	36	53	70	86	80
$H^7(T_\Phi)$	43	30	43	52	47	44	74	71	101	85
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	$\phi_{84}^{12}$	$\phi_{84}^{15}$	$\phi_{105}^5$	$\phi_{105}^{26}$	$\phi_{105}^{12}$	$\phi_{105}^6$	$\phi_{105}^{15}$	$\phi_{120}^{21}$	$\phi_{120}^4$	$\phi_{120}^{25}$
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^2(T_\Phi)$	0	0	1	0	0	1	0	0	2	0
$H^3(T_\Phi)$	1	0	4	0	2	7	0	0	9	0
$H^4(T_\Phi)$	9	5	15	1	14	27	5	1	33	0
$H^5(T_\Phi)$	50	27	53	29	63	78	34	20	99	19
$H^6(T_\Phi)$	127	78	122	113	154	160	101	94	194	99
$H^7(T_\Phi)$	117	108	134	137	147	159	133	124	185	136
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	$\phi_{168}^6$	$\phi_{168}^{21}$	$\phi_{189}^{22}$	$\phi_{189}^{20}$	$\phi_{189}^5$	$\phi_{189}^7$	$\phi_{189}^{17}$	$\phi_{210}^{10}$	$\phi_{210}^{10}$	$\phi_{210}^{25}$
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^2(T_\Phi)$	1	0	0	0	1	0	0	0	0	2
$H^3(T_\Phi)$	7	0	0	0	6	4	0	5	4	13
$H^4(T_\Phi)$	36	2	5	7	25	23	6	33	32	51
$H^5(T_\Phi)$	128	35	61	73	90	86	55	125	128	157
$H^6(T_\Phi)$	267	145	215	233	216	205	182	277	295	326
$H^7(T_\Phi)$	249	200	255	251	239	243	226	276	307	313
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	$\phi_{210}^{13}$	$\phi_{210}^{21}$	$\phi_{216}^{16}$	$\phi_{216}^9$	$\phi_{280}^8$	$\phi_{280}^{17}$	$\phi_{280}^{18}$	$\phi_{280}^9$	$\phi_{315}^{16}$	$\phi_{315}^7$
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^2(T_\Phi)$	0	0	0	0	2	0	0	0	0	0
$H^3(T_\Phi)$	0	0	1	3	9	0	0	4	0	4
$H^4(T_\Phi)$	9	2	13	21	47	7	19	28	21	31
$H^5(T_\Phi)$	68	45	99	87	191	73	126	121	141	136
$H^6(T_\Phi)$	214	185	287	224	427	257	351	306	393	347
$H^7(T_\Phi)$	253	248	296	275	404	343	388	345	441	385
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	$\phi_{336}^{14}$	$\phi_{336}^{11}$	$\phi_{378}^{14}$	$\phi_{378}^9$	$\phi_{405}^{15}$	$\phi_{405}^8$	$\phi_{420}^{10}$	$\phi_{420}^{13}$	$\phi_{512}^{12}$	$\phi_{512}^{11}$
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^2(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^3(T_\Phi)$	2	2	2	2	0	11	7	0	6	2
$H^4(T_\Phi)$	33	26	33	28	12	73	61	20	61	29
$H^5(T_\Phi)$	179	128	188	150	118	268	258	139	290	180
$H^6(T_\Phi)$	456	344	498	405	397	588	598	417	710	524
$H^7(T_\Phi)$	468	416	533	467	480	598	602	510	731	627

TABLE 4. The cohomology groups of the complement of the toric arrangement associated to  $E_7$  as representations of the Weyl group.

## REFERENCES

- [1] Arnold, V.I., *The cohomology ring of the colored braid group*, Mat. Zametki, 1969, vol. 5, issue 2, pp. 227–231.
- [2] Bergvall, O., *Cohomology of  $\mathcal{M}_3[2]$  and  $\mathcal{M}_{3,1}[2]$* , in preparation.
- [3] Brieskorn, E., *Sur les groupes de tresses*, Séminaire N. Bourbaki, 1971-1972, no. 401, pp. 21–44.
- [4] Brown, K.S., *Complete Euler Characteristics and Fixed-Point Theory*, Journal of Pure and Applied Algebra, vol. 24, issue 2, 1982, pp. 103–121.
- [5] Carter, R., *Finite Groups of Lie Type*, John Wiley and Sons, 1985.
- [6] Cox, D., Little, J., Schenck, H., *Toric Varieties*, Graduate Studies in Mathematics, American Mathematical Society, 2011.
- [7] Dimca, A., Lehrer, G., *Purity and Equivariant Weight Polynomials*, Algebraic Groups and Lie Groups, Australian Mathematical Society Lecture Notes, vol. 9, 1997, pp. 161–181.
- [8] Eisenbud, D., Sturmfels, B., *Binomial Ideals*, Duke Math. Journal, 1996, vol. 84, No. 1, pp. 1–45.
- [9] Felder, G., Veselov, A.P., *Coxeter Group Actions on the Complement of Hyperplanes and Special Involutions*, J. Europ. Math. Soc., vol. 7, issue 1, pp. 101–116.
- [10] Fleischmann, P., Janiszczak, I., *Combinatorics and Poincaré Polynomials of Hyperplane Complements for Exceptional Weyl Groups*, Journal of Combinatorial Theory, vol. 63, issue 2, pp. 257–274.
- [11] Getzler, E., Looijenga, E., *The Hodge Polynomial of  $\overline{\mathcal{M}}_{3,1}$* , preprint available at arXiv: /9910174v1, 1999.
- [12] Lehrer, G., *On the Poincaré Series Associated with Coxeter Group Actions on Complements of Hyperplanes*, J. London Math. Soc., 1987, vol. s2-36, issue 2, pp. 275–294.
- [13] Looijenga, E., *Cohomology of  $\mathcal{M}_3$  and  $\mathcal{M}_3^1$* , Contemporary Mathematics, vol. 150, American Mathematical Society, Mapping Class Groups and Moduli Spaces of Riemann Surfaces, 205–228.
- [14] MacMeican, C., *The Poincaré Polynomial of an MP Arrangement*, Proceedings of the AMS, 2004, vol. 132, No. 6.
- [15] Madsen, I., Tornehave, J., *From Calculus to Cohomology*, Cambridge University Press, 2001.
- [16] Moci, L., *Combinatorics and Invariants of Toric Arrangements*, Rend. Lincei Mat. e Appl., 2008, vol. 19, 293–308.
- [17] Orlik, P., Solomon, L., *Combinatorics and Topology of Complements of Hyperplanes*, Inventiones math., 1980, vol. 56, issue 2, pp. 167–189.

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